Exact correlation functions of Bethe lattice spin models in external magnetic fields

Chin-Kun Hu* and N. Sh. Izmailian†
Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan
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We develop a transfer matrix method to compute exactly the spin-spin correlation functions $\langle s_0 s_n \rangle$ of Bethe lattice spin models in the external magnetic field h and for any temperature T. We first compute $\langle s_0 s_n \rangle$ for the most general spin-S Ising model, which contains all possible single-ion and nearest-neighbor pair interactions. This general spin-S Ising model includes the spin-$\frac{1}{2}$ simple Ising model and the Blume-Emery-Griffiths (BEG) model as special cases. From the spin-spin correlation functions, we obtain functions of correlation length $\xi(T,h)$ for the simple Ising model and BEG model, which show interesting scaling and divergent behavior as $h \to 0$ and T approaches the critical temperature T_c. Our method to compute exact spin-spin correlation functions may be applied to other Ising-type models on Bethe and Bethe-like lattices.
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I. INTRODUCTION

Bethe [1] and the Bethe-like [2] lattices have been widely used in solid state and statistical physics [3–18], as they represent underlying lattices for which many problems can be solved exactly. The Bethe lattice has attracted particular interest because it usually reflects essential features of systems, even when conventional mean-field theories fail [11].

Besides thermodynamic quantities such as the magnetization, the specific heat, etc., the correlation function contains important information about a phase transition system [19], and is often studied by theoretical calculations [20–25] and experimental measurements [26–29]. It is widely believed that the singular behavior of physical quantities at the critical temperature T_c of second order phase transitions is related to the divergence of the correlation length ξ at T_c [19]. In connection with this, a knowledge of the exact form of the spin-spin correlation function is very crucial for locating phase transitions and for analytical investigations of physical phenomenon. Furthermore, for many experiments, spin-spin correlation functions are most relevant, since they are measured by standard probes such as linear response to an adiabatic or isothermal applied field, or scattering of neutrons or electromagnetic waves [26]. In the past several years, considerable progress has been achieved in the computation of the correlation function of statistical mechanical systems [20,22]. However, spin-spin correlation functions are exactly known only in a few models, including the two-dimensional Ising model in zero magnetic field and at any temperatures [21], and the planar Ising model in a magnetic field and exactly at T_c [22].

A long-standing problem of statistical mechanics is the exact solution of the spin-spin correlation function for the Ising model in an external magnetic field and at any temperature. In this paper, we develop a transfer matrix method to compute exactly the spin-spin correlation functions $\langle s_n s_0 \rangle$ for the most general spin-S model on the Bethe lattice for any temperatures T and external field h. The model includes the Ising model [30] and the Blume-Emery-Griffiths (BEG) model [31] as special cases. The correlation length $\xi(T,h)$ obtained from the spin-spin correlation function shows interesting scaling and divergent behavior as $h \to 0$ and $t \to 0$, where $t=(T-T_c)/T_c$ [32]. Our results thus solve a long-standing puzzle in the critical phenomena of the Bethe lattice Ising model. Our method may be applied easily to other Bethe lattice spin models.

Although the free energy of the Cayley tree Ising model in zero external magnetic field is an analytic function of the temperature, the magnetization m and the magnetic susceptibility χ of the central spin s_0 of the Bethe lattice Ising model have singular behaviors with the critical exponents $\beta=\frac{1}{2}$, $\delta=3$, and $\alpha=0$ [30]. However, there is no previous calculation which shows that ξ of the Bethe lattice Ising model has a singular behavior. Our work shows clearly that ξ diverges at the critical point of m and χ for the Bethe lattice Ising model.

The difference between the Cayley tree and the Bethe lattice was discussed by Baxter [30]. In the Cayley tree, the surface plays a very important role because the sites on the surface comprise a finite fraction of the total sites even in the thermodynamic limit. As a consequence, the spin models on the Cayley tree exhibit quite unusual types of phase transitions without long-range order [23–25]; the calculated correlation functions do not show a singular behavior [23–25]. To overcome this problem, one usually considers only properties of sites deep in the interior (away from the surface) of the Cayley tree. The union of such equivalent sites, with the same coordination number q, can be regarded as forming the Bethe lattice [30]. Thus the Bethe lattice is assumed to have translational symmetry like any regular lattice.

In this paper, we demonstrate the crucial role of Bethe lattice dimensionality in determining the critical behavior of the correlation length, and show clearly that correlation length ξ diverges at the critical point of m and χ for the Bethe lattice Ising model with critical exponent $\nu=1$, which

*Electronic address: huck@phys.sinica.edu.tw; http://www.sinica.edu.tw/~statphys/
†On leave from Theoretical Division, Yerevan Physics Institute, Yerevan, 375036, Armenia. Electronic address: izmailan@phys.sinica.edu.tw

is different from the mean-field critical exponent $\nu=\frac{1}{2}$ considered by Tsallis and Magalhães in a recent review paper [10], but is consistent with the critical exponent of the localization length associated with density-density correlator in the Bethe lattice Anderson model obtained by a supersymmetry method [8]. Our result gives independent support to the idea that the mean-field approximation and Bethe lattice approach are not equivalent in principle [11]. In this paper we also analyze exact spin-spin correlation function for the BEG model [31], and find that at the tricritical point, $\xi \sim t^{-1/2}$, with the tricritical exponent $\nu_t=\frac{1}{2}$.

Thus we have solved a long-standing puzzle in the critical phenomena of the Bethe lattice Ising model. Our approach can be extended easily to other models (Potts, Ashkin-Teller, etc.) on Bethe and Bethe-like lattices. In particular, our results for the Ising model on the Bethe lattice can be extended easily to Husimi lattices, because they can be related with each other in terms of the star-triangle transformation [30].

The outline of the paper is as follows: In Sec. II the most general spin-S model is defined, and the spin-spin correlation function of this model is evaluated in a closed form. In Sec. III we analyze the exact correlation function of the Bethe lattice spin-$\frac{1}{2}$ Ising model and discuss the critical properties. In Sec. IV the critical behavior of the spin-spin correlation function is analyzed for the spin-1 Ising model on the Bethe lattice. In Sec. V a brief discussion of our results is presented.

II. MOST GENERAL SPIN-*S* MODEL

Ising-type models with a spin greater than $\frac{1}{2}$ have rich fixed-point structures. The great interest in these models arises partly from the unusually rich phase transition behavior they display as their interaction parameters are varied, and partly from their many possible applications.

A spin-1 Ising model was initially introduced by BEG [31] in connection with phase separation and superfluid ordering in ^{3}He- ^{4}He mixtures. The BEG model has played an important role in the development of the theory of multicritical phenomena associated with various physical systems [33], and has been extensively investigated in the literature [6,7].

The spin-$\frac{3}{2}$ Ising model with dipolar and quadrupolar interactions was first introduced to explain phase transitions in $DyVO_4$ by Sivardiere and Blume [34], and a different spin-$\frac{3}{2}$ Ising model for ethanol-water-carbon-dioxide was introduced by Krinsky and Mukamel [35]. Another spin-$\frac{3}{2}$ Ising model was investigated by Barreto and De Alcantra Bonfim [36].

Let us define the most general spin-S model by the Hamiltonian

$$-\beta H=\sum_{\langle ij\rangle} H_1(s_i,s_j)+\sum_i H_2(s_i), \tag{1}$$

where $\beta=(k_BT)^{-1}$, and s_i is a spin variable which takes a value on $\{-S,-S+1,\ldots,S-1,S\}$. The first sum goes over all nearest-neighbor pairs of the Bethe lattice, and the second over all sites. $H_1(s_i,s_j)$ contains all possible nearest-neighbor pair interactions, and can be written as

$$H_1(s_i,s_j)=\sum_{\mu,\nu=1}^{2S} J_{\mu\nu}s_i^{\mu}s_j^{\nu}. \tag{2}$$

$H_2(s_i)$ includes all possible single ion interactions:

$$H_2(s_i)=\sum_{\mu=1}^{2S} h_\mu s_i^\mu. \tag{3}$$

There are $S(2S+1)$ independent nearest-neighbor coupling constants $(J_{\mu\nu})$ in Eq. (2), and $2S$ external fields h_μ in Eq. (3). Hamiltonian (1) can describe a variety of different models. The partition function has the form

$$Z=\sum_{\{s\}} \exp\left\{\sum_{\langle ij\rangle} H_1(s_i,s_j)+\sum_i H_2(s_i)\right\}. \tag{4}$$

The advantage of the Bethe lattice is that, for models formulated on it, exact recursion relations can be derived. The calculation on a Bethe lattice is done recursively [30]. When the Bethe lattice is "cut" apart at the central site 0, it separates into q identical branches, each of which contains $q-1$ branches. Then the partition function of the model can be written as

$$Z_N=\sum_s \exp(-\beta H)=\sum_{s_0} \exp\{H_2(s_0)\}g_N^q(s_0), \tag{5}$$

where s_0 is a spin in the central site, N is the number of generations ($N\to\infty$ corresponds to the thermodynamic limit), and $g_N(s_0)$ is in fact the partition function of one branch. Each branch, in turn, can be cut along any site of the first generation which is nearest to the central site. The expression for $g_N(s_0)$ can therefore be written in the form

$$g_N(s_0)=\sum_{s_1} \exp\{H_1(s_0,s_1)+H_2(s_1)\}g_{N-1}^{\gamma}(s_1), \tag{6}$$

where $\gamma=q-1$.

Consequently, we have $2S+1$ recursion relations for $g_N(s_0)$, where s_0 takes values $(-S,-S+1,\ldots,S-1,S)$. After dividing each recursion relation by the recursion relation for $g_N(S)$, we have $2S$ recursion relations for $x_N(s_0)$:

$$x_N(s_0)=\frac{\sum_{s_1} \exp\{H_1(s_0,s_1)+H_2(s_1)\}x_{N-1}^{\gamma}(s_1)}{\sum_{s_1} \exp\{H_1(S,s_1)+H_2(s_1)\}x_{N-1}^{\gamma}(s_1)}, \tag{7}$$

where

$$x_N(s_0)=g_N(s_0)/g_N(S) \tag{8}$$

and the equation for $g_N(S)$ is

$$g_N(S)=g_{N-1}^{\gamma}(S)\sum_{s_1} \exp\{H_1(S,s_1)+H_2(s_1)\}x_{N-1}^{\gamma}(s_1). \tag{9}$$

Since the right-hand side of Eq. (7) is bounded by x_N, it follows that x_N is finite for $N\to\infty$. Through x_N, obtained by Eq. (7), one can express the density $m_\mu=\langle(s_0/S)^\mu\rangle$ of central site (the symbol $\langle \cdots \rangle$ denotes the thermal average), where μ take values from 1 to $2S$:

$$m_\mu = \left\langle \left(\frac{s_0}{S} \right)^\mu \right\rangle = \frac{1}{S^\mu} \frac{\sum_{s_0} s_0^\mu \exp\{H_2(s_0)\} x_N^q(s_0)}{\sum_{s_0} \exp\{H_2(s_0)\} x_N^q(s_0)}, \quad (10)$$

and other thermodynamic parameters.

So we can say that x_N in the thermodynamic limit ($N \to \infty$) determines the states of the system. For this reason the recursion relations for x_N given by Eq. (7) can be called the equations of state for the spin-S model on the Bethe lattice. For example, at high temperatures the recursion equation (7) tends to a fixed point, and therefore the system has an appointed magnetization m. Equations (7) and (10) are fundamental equations for the spin-S model on the Bethe lattice.

For calculating the spin-spin correlation function, it is now convenient to write down the expression for partition function in the following form

$$Z_N = \sum_{s_0 s_1, \dots, s_n} \exp\left(\sum_{i=0}^{n-1} H_1(s_i, s_{i+1}) + \sum_{i=0}^{n} H_2(s_i) \right) \times g_N^\gamma(s_0) g_{N-1}^{\gamma-1}(s_1) \dots g_{N-n+1}^{\gamma-1}(s_{n-1}) g_{N-n}^\gamma(s_n), \quad (11)$$

where n denotes the number of steps from the central point 0. Summing over $s_n, s_{n-1}, \dots, s_1$ consistently, we obtain Eq. (5) again. Then the two-spin correlation function between s_0 and s_n, $\Gamma(n) = (1/S^2)\langle s_0 s_n \rangle$, can be written as

$$\Gamma(n) = \frac{1}{S^2} \frac{\sum_{s_0 s_1, \dots, s_n} s_0 s_n \exp\left(\sum_{i=0}^{n-1} H_1(s_i, s_{i+1}) + \sum_{i=0}^{n} H_2(s_i) \right) x_N^\gamma(s_0) x_{N-1}^{\gamma-1}(s_1) \cdots x_{N-n+1}^{\gamma-1}(s_{n-1}) x_{N-n}^\gamma(s_n)}{\sum_{s_0 s_1, \dots, s_n} \exp\left(\sum_{i=0}^{n-1} H_1(s_i, s_{i+1}) + \sum_{i=0}^{n} H_2(s_i) \right) x_N^\gamma(s_0) x_{N-1}^{\gamma-1}(s_1) \cdots x_{N-n+1}^{\gamma-1}(s_{n-1}) x_{N-n}^\gamma(s_n)}. \quad (12)$$

We will show below that the calculation of $\langle s_0 s_n \rangle$ can be performed by a transfer matrix method. Other techniques for the calculation of the spin-spin correlation function are not able to give the correlation in the presence of an external magnetic field. For example, the results obtained in Refs. [23–25] are appropriate only for h strictly equal to zero without any symmetry breaking effects. It should be noted that in order to obtain results which are relevant for the Bethe lattice, we must use the proper thermodynamic limit $N \to \infty$.

The properties of the Bethe lattice can be investigated by considering a Cayley tree with a very large number of generations (N), and one looks only at the thermal ensemble of the sites in the interior part of the first n generations. Then the limit $N \to \infty$ is taken before $n \to \infty$.

We are interested in the case when the series of solutions of recursion relations given by Eq. (7) converges to a stable point as $N \to \infty$. In the thermodynamic limit ($N \to \infty$), we may expect that x_{N-n} does not depend on n, so that x_{N-n} and x_N can be regarded as the same fixed-point solutions x of recursion relations given by Eq. (7), which corresponds to the behavior in the interior part of an infinite Cayley tree (i.e., the Bethe lattice). In this case

$$\lim_{N \to \infty} x_{N-n}(s) = x(s)$$

for all finite n. Then the recursion equations (equations of state) becomes

$$x(s_0) = \frac{\sum_{s_1} \exp\{H_1(s_0, s_1) + H_2(s_1)\} x^\gamma(s_1)}{\sum_{s_1} \exp\{H_1(S, s_1) + H_2(s_1)\} x^\gamma(s_1)}, \quad (13)$$

and the spin-spin correlation function $\Gamma(n)$ given by Eq. (12) takes the form

$$\Gamma(n) = \frac{1}{S^2} \frac{\sum_{s_0 s_1, \dots, s_n} s_0 s_n \exp\left(\sum_{i=0}^{n-1} H_1(s_i, s_{i+1}) + \sum_{i=1}^{n} H_2(s_i) \right) x^\gamma(s_0) x^{\gamma-1}(s_1) \cdots x^{\gamma-1}(s_{n-1}) x^\gamma(s_n)}{\sum_{s_0 s_1, \dots, s_n} \exp\left(\sum_{i=0}^{n-1} H_1(s_i, s_{i+1}) + \sum_{i=1}^{n} H_2(s_i) \right) x^\gamma(s_0) x^{\gamma-1}(s_1) \cdots x^{\gamma-1}(s_{n-1}) x^\gamma(s_n)}. \quad (14)$$

The correlation function of Eq. (14) can be expressed in the vector-matrix form. For this purpose, let us introduce a $(2S+1)\times(2S+1)$ matrix **V** and a $(2S+1)$-component column vector **R**. The elements of **V** are given by

$$V_{ss'}=\exp\left(H_1(s,s')+\frac{H_2(s)+H_2(s')}{2}\right)[x(s)x(s')]^{(\gamma-1)/2}, \tag{15}$$

where s and s' independently take values $S,S-1,\ldots,-S+1,-S$. The vector **R** and transposed vector $\mathbf{R}^T$ have elements

$$r_s=\exp\left(\frac{H_2(s)}{2}\right)x^{(\gamma+1)/2}(s). \tag{16}$$

Let us also introduce the diagonal matrix **S**

$$\mathbf{S}=\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & \frac{S-1}{S} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -1\end{pmatrix}.$$

With these definitions, we may rewrite Eq. (14) in the vector-matrix form

$$\Gamma(n)=\frac{\mathbf{R}^T\mathbf{S}\mathbf{V}^n\mathbf{S}\mathbf{R}}{\mathbf{R}^T\mathbf{V}^n\mathbf{R}}. \tag{17}$$

The transfer matrix **V** is real symmetric ($V_{ss'}=V_{s's}$), and can be diagonalized by the transformation

$$\mathbf{P}^{-1}\mathbf{V}\mathbf{P}=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{2S+1}\end{pmatrix},$$

where **P** is a $(2S+1)\times(2S+1)$ matrix with the elements $p_{ss'}$ and λ_1, λ_2, $\ldots$, and λ_{2S+1} are the eigenvalues of the matrix **V**. These eigenvalues can be obtained from the characteristic equation. Then the spin-spin correlation function of Eq. (17) can be written as

$$\Gamma(n)=\frac{\mathbf{R}^T\mathbf{S}\mathbf{P}\mathbf{L}\mathbf{P}^{-1}\mathbf{S}\mathbf{R}}{\mathbf{R}^T\mathbf{P}\mathbf{L}\mathbf{P}^{-1}\mathbf{R}}, \tag{18}$$

where **L** is a $(2S+1)\times(2S+1)$ diagonal matrix

$$\mathbf{L}=\begin{pmatrix} \lambda_1^n & 0 & \cdots & 0\\ 0 & \lambda_2^n & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{2S+1}^n\end{pmatrix}.$$

After some algebraic manipulations, using Eqs. (13), (15), and (16), we may write the correlation function of Eq. (18) as

$$\Gamma(n)=m_1^2+\sum_{k=1}^{2S} A_k l_k^n, \tag{19}$$

where $l_k\equiv\lambda_{k+1}/\lambda_1$. An explicit expression for A_k is given in the Appendix. It should be noted that the obtained exact expression for the spin-spin correlation function depends on the ratios of the eigenvalues λ_k/λ_1. Thus Eq. (19), together with Eqs. (10) and (13), give us full set of equations for investigation spin-*S* model on the Bethe lattice. In the following we turn to various examples.

III. SPIN-$\frac{1}{2}$ ISING MODEL

We first consider a spin-$\frac{1}{2}$ Ising Hamiltonian at a temperature T and an external magnetic field h,

$$-\beta H=4J\sum_{\langle ij\rangle} s_i s_j+2h\sum_i s_i, \tag{20}$$

where $s_i=+\frac{1}{2}$ or $-\frac{1}{2}$ and the first term describes the ferromagnetic coupling $(4J)$ between the spin at site i and j.

For the magnetization $(m_1=2\langle s_0\rangle)$ of the spin in the central site, from Eq. (10) we can obtain the following expression:

$$m_1=\frac{\exp(2h)-x^q}{\exp(2h)+x^q}, \tag{21}$$

where x is the fixed point of the recursion relations (13) in the thermodynamic limit

$$x^{q-1}\exp(-2h)=\frac{x\,\exp(2J)-1}{\exp(2J)-x}. \tag{22}$$

The two-spin correlation function for the spin-$\frac{1}{2}$ Ising model can be obtained from Eq. (19),

$$\Gamma(n)=m_1^2+A_1\left(\frac{\lambda_2}{\lambda_1}\right)^n, \tag{23}$$

with $A_1=1-m_1^2$ (see the Appendix), and λ_1 and λ_2 are the eigenvalues of the 2×2 matrix **V**.

The elements of **V** are given by

$$V_{ss'}=\exp(4Jss'+hs+hs')[x(s)x(s')]^{(\gamma-1)/2}, \tag{24}$$

where s and s' independently take values ±1:

$$\mathbf{V}=\begin{pmatrix} V_{++} & V_{+-}\\ V_{-+} & V_{--}\end{pmatrix}.$$

The eigenvalues of the matrix **V** can be obtained from the characteristic equation

$$\lambda^2-\lambda(V_{++}+V_{--})+V_{++}V_{--}-V^2=0. \tag{25}$$

Using Eqs. (22) and (24), we find that

$$\lambda_1 = 2\sinh(2J)\frac{\exp(J+h)}{[\exp(2J)-x]},$$
$$\lambda_2 = [2\cosh(2J)-x-x^{-1}]\frac{\exp(J+h)}{[\exp(2J)-x]}. \tag{26}$$

Thus the spin-spin correlation function can be expressed exactly as

$$\Gamma(n) = m_1^2 + (1-m_1^2)\lambda^n, \tag{27}$$

where

$$\lambda = \frac{\lambda_2}{\lambda_1} = \frac{2\cosh(2J)-x-x^{-1}}{2\sinh(2J)}, \tag{28}$$

m_1 is the magnetization of the spin in the central site given by Eq. (21), and x is the solution of the recursion relation given by Eq. (22).

It is well known that the Ising model on the Bethe lattice exhibits ferromagnetism, with a critical point at $h=0$, $x=1$, and $T=T_c$, where $J_c=\frac{1}{2}\ln[q/(q-2)]$, and critical exponents β, δ, and α have the ''classical'' values $\beta=\frac{1}{2}$, $\delta=3$, and $\alpha=0$ [30]. Let us now consider the general behavior of the spin-spin correlation function in the critical region.

First consider the cases $h=0$ and $T=T_c$. From Eq. (27), we obtain $\Gamma(n)=(q-1)^{-n}$. Let us consider a Bethe lattice with a coordination number q, whose dimension is defined by $d_n=(\ln C_n/\ln n)$ which tends to infinity with $n\to\infty$ for $q>2$ and equals 1 for $q=2$, where $C_n=[q(q-1)^n-2]/(q-2)$ is the total number of sites. We should note that for $q=2$ the Bethe lattice becomes the ordinary one-dimensional chain. In the limit of large n, d_n for all $q>2$ becomes

$$d = \frac{n}{\ln n}\ln(q-1).$$

Thus we can write, for large n,

$$\Gamma(n) = (q-1)^{-n} = n^{-d}. \tag{29}$$

Near the critical point, setting, as usual, $t=(T-T_c)/T_c$, we find that the spin-spin correlation function is

$$\Gamma(n) = \frac{\exp\left(-\frac{n}{\xi}\right)}{n^d}, \tag{30}$$

where the correlation length ξ is given by

$$\xi = \left[\ln\frac{1}{(q-1)\lambda}\right]^{-1} = \left[\ln\left(\frac{1}{q-1}\coth\frac{J_c}{1+t}\right)\right]^{-1}$$
$$\sim \frac{q-1}{q(q-2)J_c}t^{-1}. \tag{31}$$

Thus, we find that the correlation length ξ increases as the critical point is approached according to $\xi\sim t^{-\nu}$, with the critical exponent $\nu=1$. It is interesting to note that the correlation length ξ shows interesting scaling and singular behavior near the critical point. While the Ising model on the Bethe lattice exhibits, in general, a mean-field-like phase transition with ''classical'' exponents, the critical behavior of the correlation length near the transition point coincides with the correlation length behavior in a one-dimensional chain with a critical exponent $\nu=1$, which differs from its ''classical value'' $\nu=\frac{1}{2}$ [10]. The similar behavior of the localization length associated with the density-density correlator can be observed in the Anderson model on the Bethe lattice [8].

If h and t are both sufficiently small in the critical region, then, based on Eqs. (22), (28), and (31), the general behavior of the correlation length ξ should be described by a scaling function F

$$\xi = t^{-1}F(ht^{-3/2}), \tag{32}$$

where $F(x)=(f_1+f_2x^{2/3})^{-1}$, with

$$f_1 = \frac{q(q-2)}{2(q-1)}\ln\frac{q}{q-2} \quad \text{and} \quad f_2^3 = 9\frac{q(q-2)}{(q-1)^2}.$$

For a small $h>0$ and $t\to 0$, from Eq. (32) we obtain the result

$$\xi = f_2^{-1}h^{-2/3}, \tag{33}$$

i.e., the critical exponent is $\frac{2}{3}$.

The bulk susceptibility per lattice site χ or the linear response against the field is derived from Eq. (21) as $\chi=\partial m_1/\partial h$

$$\chi = \frac{\partial m_1}{\partial h} = \chi_0\left[1-\frac{(q-1)[2\cosh(2J)-x-x^{-1}]}{2\sinh(2J)}\right]^{-1}$$
$$= \frac{\chi_0}{1-(q-1)\lambda}, \tag{34}$$

where χ_0 is nonsingular part of the magnetic susceptibility, and is given by

$$\chi_0 = 2e^{-2J}\frac{[2\cosh(2J)-x-x^{-1}][2\exp(2J)-x-x^{-1}]}{\sinh(2J)[x+x^{-1}-2\exp(-2J)]^2}$$
$$= (1-m_1^2)(1+\lambda). \tag{35}$$

Thus, the magnetic susceptibility χ can finally be written in the simple form

$$\chi = \frac{(1-m_1^2)(1+\lambda)}{1-(q-1)\lambda}. \tag{36}$$

By means of the fluctuation relation $\chi=\Sigma(\Gamma(n)-m_1^2)$, we can recover Eq. (36) through Eq. (27). To prove this statement let us consider in more detail the fluctuation relation

$$\chi = \lim_{N\to\infty}\frac{1}{N_s}\sum_{ij}\frac{1}{S^2}(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle), \tag{37}$$

where the sum goes over all pairs of sites on the Bethe lattice, and N_s is the total number of sites. To carry out the summations in Eq. (37), we first note that by definition all sites on the Bethe lattice are equivalent, and consequently,

$$\langle s_i\rangle=\langle s_0\rangle,$$

$$\sum_{ij}\langle s_is_j\rangle=N_s\sum_j\langle s_0s_j\rangle.$$

Using these relations we may rewrite Eq. (37) as

$$\chi=\lim_{N\to\infty}\sum_j\frac{1}{S^2}(\langle s_0s_j\rangle-\langle s_0\rangle^2)$$
$$=\lim_{n\to\infty}\sum_{j=1}^{n}c_j[\Gamma(j)-m_1^2], \tag{38}$$

where c_j is the number of sites which is j steps away from the central site 0,

$$c_j=q(q-1)^{j-1}\quad(c_0=1), \tag{39}$$

$\Gamma(j)$ is the spin-spin correlation function given by Eq. (27), and $m_1=\langle s_0\rangle$ is the magnetization of the spin in the central site.

Substituting Eqs. (27) and (39) into Eq. (38), we find

$$\chi=\lim_{n\to\infty}(1-m_1^2)\left[1+\sum_{j=1}^{n}q(q-1)^{j-1}\lambda^j\right]$$
$$=\lim_{n\to\infty}(1-m_1^2)\left\{\frac{1+\lambda}{1-(q-1)\lambda}-q\lambda\frac{[(q-1)\lambda]^{n-2}}{1-(q-1)\lambda}\right\}. \tag{40}$$

It is now clear that the susceptibility χ diverges for $(q-1)\lambda\geqslant1$, and given by Eq. (36) for $(q-1)\lambda<1$. From Eq. (36) we can easily establish the relation between the susceptibility χ and the correlation length ξ in the critical region

$$\chi\sim\xi.$$

IV. SPIN-1 ISING MODEL

Let us consider, for example, a spin-1 Ising model, which is known as the Blume-Emery-Griffiths (BEG) model [31]. The BEG model on the Bethe lattice was studied in Refs. [6,7]. The model has played an important role in the development of the theory of tricritical phenomena [33].

The Hamiltonian of the spin-1 Ising model on the Bethe lattice is given by

$$-\beta H=J\sum_{\langle ij\rangle}s_is_j+K\sum_{\langle ij\rangle}s_i^2s_j^2-\Delta\sum_i s_i^2+h\sum_i s_i, \tag{41}$$

where $\beta=(k_BT)^{-1}$ and $s_i=+1,0,-1$ is the spin variable at site i. The first term describes the ferromagnetic coupling (J) between the spin at sites i and j, and the second term describes the biquadratic coupling (K). Both interactions are restricted to the q nearest-neighbor pairs of spins. The third term describes the single ion anisotropy Δ, and the last term represents the effects of an external magnetic field (h).

This model has two order parameters: one is the thermal average of the total spin $m_1=\langle s_0\rangle$ and the other is the quadrupolar moment $m_2=\langle s_0^2\rangle$ which reflects the possibility of phase separation. These order parameters are expressed by

$$m_1=\frac{\exp(h-\Delta)y^q-\exp(-h-\Delta)x^q}{1+\exp(h-\Delta)y^q+\exp(-h-\Delta)x^q}, \tag{42}$$

$$m_2=\frac{\exp(h-\Delta)y^q+\exp(-h-\Delta)x^q}{1+\exp(h-\Delta)y^q+\exp(-h-\Delta)x^q}. \tag{43}$$

From Eq. (12) we can obtain the following expression for the first-neighbor spin-spin correlation function $\langle s_0s_1\rangle$

$$\langle s_0s_1\rangle=\frac{(e^{2h}y^{2\gamma}+e^{-2h}x^{2\gamma})e^{(J+K-2\Delta)}-2x^\gamma y^\gamma e^{(-J+K-2\Delta)}}{1+2(e^hy^\gamma+e^{-h}x^\gamma)e^{-\Delta}+(e^{2h}y^{2\gamma}+e^{-2h}x^{2\gamma})e^{(J+K-2\Delta)}-2x^\gamma y^\gamma e^{(-J+K-2\Delta)}} \tag{44}$$

where

$$x=\lim_{N\to\infty}\frac{g_N(-)}{g_N(0)}\quad\text{and}\quad y=\lim_{N\to\infty}\frac{g_N(+)}{g_N(0)}.$$

Let us introduce the new variables

$$v=\frac{x-y}{2}\quad\text{and}\quad u=\frac{x+y-2}{2};$$

then we obtain

$$m_1=-v\frac{u(1+a)+a}{b+u^2+av^2}, \tag{45}$$

$$m_2=\frac{u(u+1)+av^2}{b+u^2+av^2}, \tag{46}$$

and

$$\langle s_0s_1\rangle=\frac{bu^2+a^3(b+1)v^2}{ab(b+u^2+av^2)}, \tag{47}$$

where u and v are the solution of the recursion relation given by Eq. (13) in the thermodynamic limit ($N \to \infty$):

$$\exp(2h) = \frac{u - av}{u + av}\left(\frac{u+1+v}{u+1-v}\right)^{q-1}, \tag{48}$$

$$\exp(2\Delta) = \frac{4(b-u)^2}{u^2 - a^2 v^2}[(u+1)^2 - v^2]^{q-1}, \tag{49}$$

and b and a are the following constants:

$$b = \exp(K)\cosh(J) - 1,$$

$$a = \frac{\exp(K)\cosh(J) - 1}{\exp(K)\sinh(J)}.$$

It follows from Eq. (19) that the spin-spin correlation function for spin-1 Ising model can be written as

$$\Gamma(n) = m_1^2 + \sum_{k=1}^{2} A_k \left(\frac{\lambda_{k+1}}{\lambda_1}\right)^n, \tag{50}$$

where A_1 and A_2 take the forms (see the Appendix)

$$A_1 = \frac{\lambda_3(m_2 - m_1^2) - \lambda_1(\langle s_0 s_1\rangle - m_1^2)}{\lambda_3 - \lambda_2}$$

and

$$A_2 = \frac{\lambda_2(m_2 - m_1^2) - \lambda_1(\langle s_0 s_1\rangle - m_1^2)}{\lambda_2 - \lambda_3},$$

with m_1, m_2, and $\langle s_0 s_1\rangle$ given by Eqs. (45), (46), and (47), respectively. The eigenvalues λ_1, λ_2, and λ_3 of the symmetric 3×3 matrix $\mathbf{V}$

$$\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{12} & V_{22} & V_{23} \\ V_{13} & V_{23} & V_{33} \end{pmatrix},$$

can be obtained from the characteristic equation

$$(V_{11}-\lambda)(V_{22}-\lambda)(V_{33}-\lambda) + 2V_{12}V_{13}V_{23} = V_{23}^2(V_{11}-\lambda) + V_{13}^2(V_{22}-\lambda) + V_{12}^2(V_{33}-\lambda).$$

The elements of the matrix $\mathbf{V}$ are given by

$$V_{ss'} = \exp\left(Jss' + Ks^2 s'^2 - \Delta\frac{s^2 + s'^2}{2} + h\frac{s+s'}{2}\right) \times [x(s)x(s')]^{(\gamma-1)/2}, \tag{51}$$

where s and s' may independently take values $+1,\ 0,$ and -1.

Using Eqs. (48), (49), and (51), we find that

$$\lambda_1 = \frac{b}{b-u}, \quad \lambda_{2,3} = (C \pm \sqrt{C^2 - D})\lambda_1, \tag{52}$$

where

$$C = -\frac{u}{2b} + \frac{(ab+a+b)(u^2+u-av^2)}{2ab[(u+1)^2 - v^2]},$$

and

$$D = \frac{(b-u)(u^2 - a^2 v^2)}{ab[(u+1)^2 - v^2]}.$$

The global phase diagram of the spin-1 Ising model on the Bethe lattice was studied in detail in Refs. [6,7]. The Λ line of the phase transition in the (J, K, and Δ) space is given by conditions

$$h = 0 \quad \text{and} \quad \exp(\Delta_\Lambda) = \frac{2(b-u_c)}{u_c}(u_c+1)^{q-1}, \tag{53}$$

where

$$u_c = \frac{a}{q-1-a}.$$

In terms of the T, Δ_Λ/J, and K/J, Eq. (53) of the Λ line implies a relation $T = T_c(\Delta_\Lambda/J, K/J)$, which locates the critical temperature as a function of Δ_Λ/J and K/J.

The critical line starts at $\Delta_\Lambda \to -\infty$, $T_c/J = \frac{1}{2}\ln[q/(q-2)]$, which corresponds to the critical temperature of the spin-$\frac{1}{2}$ Ising model. We note that for $\Delta \to -\infty$, the state $s_i = 0$ is suppressed, and Hamiltonian (41) reduces to the spin-$\frac{1}{2}$ Ising model with interaction J and external magnetic field h. For certain values of K/J the system possesses a tricritical point at which the phase transition changes from the second order to the first order. A tricritical point satisfies the equation [7]

$$\frac{u_c+1}{\beta - u_c} = q - 2 + \frac{q-3}{2q}\frac{1}{u_c}. \tag{54}$$

In particular,

$$\frac{J}{T_t} = \frac{1}{2}\ln\frac{q(3q-2)}{3(q-2)} \quad \text{when} \quad \frac{K}{J} = 1$$

and

$$\frac{J}{T_t} = \frac{1}{2}\ln\frac{-q+6+\sqrt{49q^2 - 36q + 36}}{6(q-2)} \quad \text{when} \quad \frac{K}{J} = 3.$$

Let us now consider the general behavior of the spin-spin correlation function in the critical region. First consider the case $h = 0$ $(v = 0)$; then we have

$$\Gamma(n) = \frac{u(u+1)}{u^2 + b}\lambda^n,$$

with

$$\lambda = \frac{\lambda_2}{\lambda_1} = \frac{u}{a(u+1)}.$$

On the critical line ($T = T_c$, $h = 0$, and $\Delta = \Delta_\Lambda$) given by Eq. (53), the spin-spin correlation function can be expressed as

$$\Gamma(n)=\frac{u_c(u_c+1)}{u_c^2+b}\left[\frac{u_c}{a(u_c+1)}\right]^n$$

$$=\frac{a(q-1)}{a^2+b(q-1-a)}(q-1)^{-n}. \tag{55}$$

By analogy with the spin-$\frac{1}{2}$ case, we can write

$$\Gamma(n)=\frac{a(q-1)}{a^2+b(q-1-a)}n^{-d}, \tag{56}$$

where d is the dimension of the Bethe lattice. For $T_c/J=\frac{1}{2}\ln[q/(q-2)]$, we obtain $\Gamma(n)=(q-1)^{-n}$, which corresponds to the case of the spin-$\frac{1}{2}$ Ising model. In the critical region, $\Gamma(n)$ has an asymptotic decay of the form

$$\Gamma(n)\sim\frac{\exp\left(-\frac{n}{\xi}\right)}{n^d}, \tag{57}$$

where ξ is the correlation length and is given by

$$\xi=\left[\ln\frac{1}{(q-1)(\lambda_2/\lambda_1)}\right]^{-1}.$$

Near the critical Λ line, setting $u-u_c=(u_c+1)\delta$, $v=(u_c+1)\varepsilon$, and $T=T_c(1+t)$, the h, Δ, and λ_2 may be expanded, for small δ, ε, and t, as

$$h=\gamma\varepsilon\left[h_0t-(a-1)\delta+\frac{(\gamma^2-1)}{3}\varepsilon^2+(a^2-1)\delta^2\right.$$
$$\left.-(\gamma^2a-1)\varepsilon^2\delta+\frac{\gamma^4-1}{5}\varepsilon^4\right], \tag{58}$$

$$\Delta-\Delta_\Lambda=-\Delta_0t+(\gamma-a-b)\delta+\gamma\frac{\gamma-1}{2}\varepsilon^2+\frac{a^2-b^2-\gamma}{2}\delta^2$$
$$-(\gamma a-1)\gamma\varepsilon^2\delta+\gamma\frac{\gamma^3-1}{4}\varepsilon^4, \tag{59}$$

where $\gamma=q-1, a=(u_c+1)/u_c$, $b=(u_c+1)/(\beta-u_c)$,

$$h_0=\frac{J_c(\exp K_c-\cosh J_c)-K_c\sinh J_c}{(\exp K_c\cosh J_c-1)\sinh J_c},$$

and

$$\Delta_0=\exp K_c\frac{J_c\sinh J_c+K_c\cosh J_c}{\exp K_c\cosh J_c-1}.$$

Consider the case when $h=0$ and $\Delta=\Delta_\Lambda$. From Eqs. (58) and (59), we obtain

$$c_1t=c_2\delta+c_3\delta^2, \tag{60}$$

with

$$c_1=\Delta_0+\frac{3\gamma}{2(\gamma+1)}h_0,\quad c_2=\gamma-1-\frac{u_c+1}{\beta-u_c}+\frac{1}{u_c}\frac{\gamma-2}{2(\gamma+1)},$$

and

$$c_3=\frac{a^2-b^2-\gamma}{2}-\frac{3\gamma}{\gamma+1}\frac{1}{u_c}-\frac{3\gamma(\gamma^2-\gamma-3)}{4(\gamma^2-1)(\gamma+1)}\frac{1}{u_c^2}.$$

It is easy to see from Eqs. (54) and (60) that in all points on the Λ line, $t\sim\delta$, except for the tricritical point, where $t\sim\delta^2$.

Using a Taylor expansion of the expression for the correlation length ξ by small ε, δ, and t, and Eqs. (54) and (60), we find that the correlation length ξ increases as the critical point is approached according to $\xi\sim t^{-1}$, with the critical exponent $\nu=1$, everywhere on the Λ line except the tricritical point, where $\xi\sim t^{-1/2}$ with the tricritical exponent $\nu_t=\frac{1}{2}$.

V. SUMMARY AND DISCUSSION

Let us now briefly summarize our results. In this paper we consider the most general spin-S model on the Bethe lattice in the external magnetic field, and use the transfer matrix approach to derive the spin-spin correlation function. The general spin-S model includes the spin-$\frac{1}{2}$ simple Ising model and BEG model as special cases. From the exact spin-spin correlation functions $\Gamma(n)$ of the Bethe lattice Ising model and BEG model in an arbitrary magnetic field h and temperature T, the correlation length ξ has been determined analytically. In the critical region the correlation length ξ of the simple Ising model is inversely proportional to the distance $(T-T_c)/T_c$ from the critical point. Such a singular behavior coincides with the correlation length behavior in the one-dimensional chain. We also obtain that near the transition point the magnetic susceptibility is proportional to the correlation length ξ.

Recently, Gujrati [11] showed that in many cases the behaviors on Bethe or Bethe-like lattices are qualitatively correct even when conventional mean-field theories fail. By a proper choice of these lattices, it is possible to satisfy frustrations, gauge symmetries, etc., which are usually lost in conventional mean-field calculations, because of the lack of correlations. Such correlations are present on the Bethe-like lattices, and in this paper we have given the exact expression for such correlations.

It should be noted that we can obtain the proper singular behavior of ξ because we have used the proper thermodynamic limit ($N\to\infty$) to obtain the recursion equations and correlation function, i.e., Eqs. (13) and (14), for the most general spin-S model on the Bethe lattice, and we have demonstrated the crucial role of Bethe lattice dimensionality in determining critical behavior of the correlation length.

In conclusion, it must be remarked that the transfer matrix methods discussed in this paper can be extended without difficulty to obtain correlation functions with singular correlation length for other spin models, e.g., Potts model, the multilayer Ising model, the Ising model with competing nearest-neighbor and next-nearest-neighbor interactions, etc., on Bethe and Bethe-like structures. This approach should be applicable for gauge models on generalized multiplaquette hierarchical structures as well.

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APPENDIX

The coefficient A_k $(k=1,2,\ldots,2S)$ can be obtained by solving the following system of linear algebraic equations

$$\begin{cases} g_1=A_1+A_2+\cdots+A_{2S}, \\ g_2=A_1 l_1+A_2 l_2+\cdots+A_{2S} l_{2S}, \\ \vdots \\ g_{2S}=A_1 l_1^{2S-1}+A_2 l_2^{2S-1}+\cdots+A_{2S} l_{2S}^{2S-1}, \end{cases}$$

where

$$g_k \equiv \langle s_0 s_{k-1}\rangle - m_1^2 \quad \text{for } k=1,2,\ldots,2S,$$

and $l_k \equiv \lambda_{k+1}/\lambda_1$.

Let us introduce the elementary Lagrange interpolation polynomials

$$L_i(l)=\prod_{\substack{j=1\\ j\neq i}}^{2S} \frac{l-l_j}{l_i-l_j}=\sum_{j=1}^{2S} a_{ij} l^{j-1}, \quad i=1,2,\ldots,2S,$$

which satisfy $L_i(l_j)=\delta_{ij}$, where δ_{ij} is the symbol Kronecer. Now, it is evident that

$$A_k=\sum_{j=1}^{2S} a_{kj} g_j, \quad k=1,2,\ldots,2S.$$

The elements of a_{kj} are

$$a_{kj}=\frac{(-1)^j F_j(k)}{P_k(l_k)},$$

where $P_k(l)$ is $2S-1$ degree polynomials defined by

$$P_k(l)=\prod_{\substack{i=1\\ i\neq k}}^{2S} (l-l_i), \quad k=1,2,\ldots,2S$$

and $F_j(k)$ is the elementary symmetric function in $2S-1$ variables $l_1,l_2,\ldots,l_{k-1},l_{k+1},\ldots,l_{2S}$:

$$F_{2S}=1,$$
$$F_{2S-1}=l_1+\cdots+l_{k-1}+l_{k+1}+\cdots+l_{2S},$$
$$\vdots$$
$$F_1=l_1\ldots l_{k-1} l_{k+1}\ldots l_{2S}.$$

Thus the coefficients A_k will take the form

$$A_k=\frac{\sum_{j=1}^{2S}(-1)^j F_j(k)}{\prod_{i=1}^{2S}(l_k-l_i)}, \quad i\neq k.$$

Examples follow.

1. $S=\frac{1}{2}$

$$A_1=g_1=\langle s_0^2\rangle - m_1^2 = 1-m_1^2. \tag{A1}$$

For the spin-$\frac{1}{2}$ Ising model, $\langle s_0^2\rangle \equiv 1$.

2. $S=1$

$$A_1=\frac{l_2 g_1-g_2}{l_2-l_1} = \frac{\lambda_3(m_2-m_1^2)-\lambda_1(\langle s_0 s_1\rangle - m_1^2)}{\lambda_3-\lambda_2} \tag{A2}$$

and

$$A_2=\frac{l_1 g_1-g_2}{l_1-l_2} = \frac{\lambda_2(m_2-m_1^2)-\lambda_1(\langle s_0 s_1\rangle - m_1^2)}{\lambda_2-\lambda_3}, \tag{A3}$$

where $m_2=\langle s_0^2\rangle$.

3. $S=\frac{3}{2}$

$$\langle s_0 s_n\rangle - m_1^2 = A_1 l_1^n + A_2 l_2^n + A_3 l_3^n,$$

with

$$A_1=\frac{l_2 l_3 g_1-(l_2+l_3)g_2+g_3}{(l_2-l_1)(l_3-l_1)}$$

and

$$A_2=A_1(l_1\Leftrightarrow l_2), \quad A_3=A_1(l_1\Leftrightarrow l_3),$$

where $g_1=\langle s_0^2\rangle - m_1^2$, $g_2=\langle s_0 s_1\rangle - m_1^2$, and $g_3=\langle s_0 s_2\rangle - m_1^2$.

4. $S=2$

$$\langle s_0 s_n\rangle - m_1^2 = A_1 l_1^n + A_2 l_2^n + A_3 l_3^n + A_4 l_4^n,$$

$$A_1=\frac{F_1 g_1 - F_2 g_2 + F_3 g_3 - F_4 g_4}{(l_2-l_1)(l_3-l_1)(l_4-l_1)},$$

with

$$F_1=l_2 l_3 l_4, \quad F_2=l_2 l_3+l_2 l_4+l_3 l_4,$$

$$F_3=l_2+l_3+l_4,$$

$$F_4=1$$

and

$$A_k=A_1(l_1\Leftrightarrow l_k) \quad \text{for } k=2,3,4,$$

where $g_1=\langle s_0^2\rangle - m_1^2$, $g_2=\langle s_0 s_1\rangle - m_1^2$, $g_3=\langle s_0 s_2\rangle - m_1^2$, and $g_4=\langle s_0 s_3\rangle - m_1^2$.

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